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Electromagnetic wave scattering by many small particles and creating materials with a desired permeability

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Abstract

Scattering of electromagnetic (EM) waves by many small particles (bodies), embedded in a homogeneous medium, is studied. Physical properties of the particles are described by their boundary impedances. The limiting equation is obtained for the effective EM field in the limiting medium, in the limit $a \to 0$, where a is the characteristic size of a particle and the number $M(a)$ of the particles tends to infinity at a suitable rate. The proposed theory allows one to create a medium with a desirable spatially inhomogeneous permeability. The main new physical result is the explicit analytical formula for the permeability $\mu(x)$ of the limiting medium. While the initial medium has a constant permeability μ_0, the limiting medium, obtained as a result of embedding many small particles with prescribed boundary impedances, has a non-homogeneous permeability which is expressed analytically in terms of the density of the distribution of the small particles and their boundary impedances. Therefore, a new physical phenomenon is predicted theoretically, namely, appearance of a spatially inhomogeneous permeability as a result of embedding of many small particles whose physical properties are described by their boundary impedances.

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1 Introduction

In this paper we outline a theory of electromagnetic (EM) wave scattering by many small particles (bodies) embedded in a homogeneous medium which is described by the constant permittivity $\epsilon_0 > 0$, permeability $\mu_0 > 0$ and, possibly, constant conductivity $\sigma_0 \geq 0$. The small particles are embedded in a finite domain Ω. The medium, created by the embedding of the small particles, has new physical properties. In particular, it has a spatially inhomogeneous magnetic permeability $\mu(x)$, which can be controlled by the choice of the boundary impedances of the embedded small particles and their distribution density. This is a new physical effect, as far as the author knows. An analytic formula for the permeability of the new medium is derived:

$$\mu(x) = \frac{\mu_0}{\Psi(x)},$$

where

$$\Psi(x) = 1 + \frac{8\pi}{3} i\epsilon_0 \omega h(x) N(x).$$

Here ω is the frequency of the EM field, ϵ_0 is the constant dielectric parameter of the original medium, $h(x)$ is a function describing boundary impedances of the small embedded particles, and $N(x) \geq 0$ is a function describing the distribution of these particles. We assume that in any subdomain Δ, the number $\mathcal{N}(\Delta)$ of the embedded particles D_m is given by the formula:

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0,$$

where $N(x) \geq 0$ is a continuous function, vanishing outside of the finite domain Ω in which small particles (bodies) D_m are distributed, $\kappa \in (0, 1)$ is a number one can choose at will, and the boundary impedances of the small particles are defined by the formula

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \quad x_m \in D_m,$$

where x_m is a point inside $m-$th particle D_m, Re $h(x) \geq 0$, and $h(x)$ is a continuous function vanishing outside Ω. The impedance boundary condition on the surface S_m of the $m-$th particle D_m is $E^t = \zeta_m[H^t, N]$, where E^t (H^t) is the tangential component of E (H) on S_m, and N is the unit normal to S_m, pointing out of D_m.

Since one can choose the functions $N(x)$ and $h(x)$, one can create a desired magnetic permeability in Ω. This is a novel idea, to the author's knowledge.

We also derive an analytic formula for the refraction coefficient of the medium in Ω created by the embedding of many small particles. An equation for the EM field in the limiting medium is derived. This medium is created when the size a of small particles tends to zero while the total number $M = M(a)$ of the particles tends to infinity at a suitable rate.

The refraction coefficient in the limiting medium is spatially inhomogeneous.

Our theory may be viewed as a "homogenization theory", but it differs from the usual homogenization theory (see, e.g., [1], [2], and references therein) in several respects: we do not assume any periodic structure in the distribution of small bodies, our operators are non-selfadjoint, the spectrum of these operators is not discrete, etc. Our ideas, methods, and techiques are quite different from the usual methods. These ideas are similar to the ideas developed in papers [4, 5], where scalar wave scattering by small bodies was studied, and in the papers [6],[7]. However, the scattering of EM waves brought new technical difficulties which are resolved in this paper. The difficulties come from the vectorial nature of the boundary conditions. Our arguments are valid for small particles of arbitrary shapes.

We also give a new numerical method for solving many-body wave-scattering problems for small scatterers, see Section 5.2.

2 EM wave scattering by many small particles

We assume that many small bodies D_m, $1 \leq m \leq M$, are embedded in a homogeneous medium with constant parameters ϵ_0, μ_0. Let $k^2 = \omega^2 \epsilon_0 \mu_0$, where ω is the frequency. Our arguments remain valid if one assumes that the medium has a constant conductivity $\sigma_0 > 0$. In this case ϵ_0 is replaced by $\epsilon_0 + i\frac{\sigma_0}{\omega}$. Denote by $[E, H] = E \times H$ the cross product of two vectors, and by $(E, H) = E \cdot H$ the dot product of two vectors.

Electromagnetic (EM) wave scattering problem consists of finding vectors E and H satisfying the Maxwell equations:

$$\nabla \times E = i\omega\mu_0 H, \quad \nabla \times H = -i\omega\epsilon_0 E \quad \text{in } D := \mathbb{R}^3 \setminus \cup_{m=1}^{M} D_m, \tag{1}$$

the impedance boundary conditions:

$$[N, [E, N]] = \zeta_m [H, N] \text{ on } S_m, \ 1 \leq m \leq M, \tag{2}$$

and the radiation conditions:

$$E = E_0 + v_E, \quad H = H_0 + v_H, \tag{3}$$

where ζ_m is the impedance, N is the unit normal to S_m pointing out of D_m, E_0, H_0 are the incident fields satisfying equations (1) in all of $\mathbb{R}^3$. One often assumes that the incident wave is a plane wave, i.e., $E_0 = \mathcal{E}e^{ik\alpha\cdot x}$, $\mathcal{E}$ is a constant vector, $\alpha \in S^2$ is a unit vector, S^2 is the unit sphere in $\mathbb{R}^3$, $\alpha \cdot \mathcal{E} = 0$, v_E and v_H satisfy the radiation condition: $r(\frac{\partial v}{\partial r} - ikv) = o(1)$ as $r := |x| \to \infty$.

By impedance ζ_m we assume in this paper either a constant, Re $\zeta_m \geq 0$, or a matrix function 2×2 acting on the tangential to S_m vector fields, such that

$$\mathrm{Re}(\zeta_m E^t, E^t) \geq 0 \quad \forall E^t \in T_m, \tag{4}$$

where T_m is the set of all tangential to S_m continuous vector fields such that $\mathrm{Div} E^t = 0$, where Div is the surface divergence, and E^t is the tangential component of E. Smallness of D_m means that $ka \ll 1$, where $a = 0.5 \max_{1\leq m\leq M} \mathrm{diam} D_m$. By the tangential to S_m component E^t of a vector field E the following is understood in this paper:

$$E^t = E - N(E, N) = [N, [E, N]]. \tag{5}$$

This definition differs from the one used often in the literature, namely, from the definition $E^t = [N, E]$. Our definition (5) corresponds to the geometrical meaning of the tangential component of E and, therefore, should be used. The impedance boundary condition is written usually as

$$E^t = \zeta[H^t, N],$$

where the impedance ζ is a number. If one uses definition (5), then this condition reduces to (2), because $[[N, [H, N]], N] = [H, N]$.

Lemma 1. *Problem* (1)-(4) *has at most one solution.*

Lemma 1 is proved in Section 2.

Let us note that problem (1)-(4) is equivalent to the problems (6), (7), (3), (4), where

$$\nabla \times \nabla \times E = k^2 E \text{ in } D, \quad H = \frac{\nabla \times E}{i\omega\mu_0}, \tag{6}$$

$$[N, [E, N]] = \frac{\zeta_m}{i\omega\mu_0}[\nabla \times E, N] \text{ on } S_m,\ 1 \leq m \leq M. \tag{7}$$

Thus, we have reduced our problem to finding one vector $E(x)$. If $E(x)$ is found, then $H = \frac{\nabla\times E}{i\omega\mu_0}$.

Let us look for E of the form

$$E = E_0 + \sum_{m=1}^{M} \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt, \quad g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{8}$$

where $t \in S_m$ and dt is an element of the area of S_m, $\sigma_m(t) \in T_m$. This E for any continuous $\sigma_m(t)$ solves equation (6) in D because E_0 solves (6) and

$$\begin{aligned}\nabla \times \nabla \times \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt &= \nabla\nabla \cdot \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt \\ &- \nabla^2 \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt \\ &= k^2 \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt, \quad x \in D.\end{aligned} \tag{9}$$

Here we have used the known identity $divcurlE = 0$, valid for any smooth vector field E, and the known formula

$$-\nabla^2 g(x,y) = k^2 g(x,y) + \delta(x-y). \tag{10}$$

The integral $\int_{S_m} g(x,t)\sigma_m(t)dt$ satisfies the radiation condition. Thus, formula (8) solves problem (6), (7), (3), (4), if $\sigma_m(t)$ are chosen so that boundary conditions (7) are satisfied.

Define the effective field $E_e(x) = E_e^m(x) = E_e^{(m)}(x,a)$, acting on the $m-$th body D_m:

$$E_e(x) := E(x) - \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt := E_e^{(m)}(x), \tag{11}$$

where we assume that x is in a neigborhood of S_m, but $E_e(x)$ is defined for all $x \in \mathbb{R}^3$. Let $x_m \in D_m$ be a point inside D_m, and $d = d(a)$ be the distance between two neighboring small bodies. We assume that

$$\lim_{a\to 0} \frac{a}{d(a)} = 0, \quad \lim_{a\to 0} d(a) = 0. \tag{12}$$

We will prove later that $E_e(x,a)$ tends to a limit $E_e(x)$ as $a \to 0$, and $E_e(x)$ is a twice continuously differentiable function. To derive an integral equation for $\sigma_m = \sigma_m(t)$, substitute

$$E = E_e + \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt$$

into (7), use the formula

$$[N, \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt]_{\mp} = \int_{S_m} [N_s, [\nabla_x g(x,t)|_{x=s}, \sigma_m(t)]]dt \pm \frac{\sigma_m(t)}{2}, \tag{13}$$

(see, e.g., [3]), the -(+) signs denote the limiting values of the left-hand side of (13) as $x \to s$ from D (D_m), and get the following equation (see Appendix):

$$\sigma_m(t) = A_m \sigma_m + f_m, \quad 1 \le m \le M. \tag{14}$$

Here A_m is a linear Fredholm-type integral operator, and f_m is a continuously differentiable function. Let us specify A_m and f_m. One has (see Appendix):

$$f_m = 2[f_e(s), N_s], \quad f_e(s) := [N_s, [E_e(s), N_s]] - \frac{\zeta_m}{i\omega\mu_0}[\nabla \times E_e, N_s]. \tag{15}$$

Condition (7) and formula (13) yield

$$\begin{aligned} & f_e(s) + \frac{1}{2}[\sigma_m(s), N_s] + [\int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt, N_s] \\ & - \frac{\zeta_m}{i\omega\mu_0}[\nabla \times \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt, N_s]|_{x \to s} = 0 \end{aligned} \tag{16}$$

Using the formula $\nabla \times \nabla \times = graddiv - \nabla^2$, the relation

$$\begin{aligned} \nabla_x \nabla_x \cdot \int_{S_m} g(x,t)\sigma_m(t)dt &= \nabla_x \int_{S_m} (-\nabla_t g(x,t), \sigma_m(t))dt \\ &= \nabla_x \int_{S_m} g(x,t) \mathrm{Div} \sigma_m(t)dt = 0, \end{aligned} \tag{17}$$

where Div is the surface divergence, and the formula

$$-\nabla_x^2 \int_{S_m} g(x,t)\sigma_m(t)dt = k^2 \int_{S_m} g(x,t)\sigma_m(t)dt, \quad x \in D, \tag{18}$$

where equation (10) was used, one gets from (16) the following equation

$$-[N_s, \sigma_m(s)] + 2f_e(s) + 2B\sigma_m = 0. \tag{19}$$

Here

$$B\sigma_m := [\int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt, N_s] + \zeta_m i\omega\epsilon_0[\int_{S_m} g(s,t)\sigma_m(t)dt, N_s]. \tag{20}$$

Take cross product of N_s with the left-hand side of (19) and use the formulas $N_s \cdot \sigma_m(s) = 0$, $f_m := f_m(s) := 2[f_e(s), N_s]$, and

$$[N_s, [N_s, \sigma_m(s)]] = -\sigma_m(s), \tag{21}$$

to get from (19) equation (14):

$$\sigma_m(s) = 2[f_e(s), N_s] - 2[N_s, B\sigma_m] := A_m\sigma_m + f_m, \tag{22}$$

where $A_m\sigma_m = -2[N_s, B\sigma_m]$. The operator A_m is linear and compact in the space $C(S_m)$, so that equation (22) is of Fredholm type. Therefore, equation (22) is solvable for any $f_m \in T_m$ if the homogeneous version of (22) has only the trivial solution $\sigma_m = 0$. In this case the solution σ_m to equation (22) is of the order of the right-hand side f_m, that is, $O(a^{-\kappa})$ as $a \to 0$, see formula (15). Moreover, it follows from equation (22) that the main term of the asymptotics of σ_m as $a \to 0$ does not depend on $s \in S_m$.

Lemma 2. *Assume that $\sigma_m \in T_m$, $\sigma_m \in C(S_m)$, and $\sigma_m(s) = A_m\sigma_m$. Then $\sigma_m = 0$.*

Lemma 2 is proved in Section 2.

Let us assume that in any subdomain Δ, the number $\mathcal{N}(\Delta)$ of the embedded bodies D_m is given by the formula:

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0, \tag{23}$$

where $N(x) \geq 0$ is a continuous function, vanishing outside of a finite domain Ω in which small bodies D_m are distributed, $\kappa \in (0, 1)$ is a number one can choose at will. We also assume that

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \quad x_m \in D_m, \tag{24}$$

where Re $h(x) \geq 0$, and $h(x)$ is a continuous function vanishing outside Ω.

Let us write (8) as

$$E(x) = E_0(x) + \sum_{m=1}^{M} [\nabla_x g(x, x_m), Q_m] + \sum_{m=1}^{M} \nabla \times \int_{S_m} (g(x, t) - g(x, x_m))\sigma_m(t)dt, \tag{25}$$

where

$$Q_m := \int_{S_m} \sigma_m(t)dt. \tag{26}$$

Since $\sigma_m = O(a^{-\kappa})$, one has $Q_m = O(a^{2-\kappa})$. We want to prove that the second sum in (25) is negligible compared with the first sum. One has

$$j_1 := |[\nabla_x g(x, x_m), Q_m]| \le O\left(\max\left\{\frac{1}{d^2}, \frac{k}{d}\right\}\right) O(a^{2-\kappa}), \tag{27}$$

$$j_2 := |\nabla \times \int_{S_m} (g(x,t) - g(x, x_m))\sigma_m(t)dt| \le aO\left(\max\left\{\frac{1}{d^3}, \frac{k^2}{d}\right\}\right) O(a^{2-\kappa}), \tag{28}$$

and

$$\left|\frac{j_2}{j_1}\right| = O\left(\max\left\{\frac{a}{d}, ka\right\}\right) \to 0, \qquad \frac{a}{d} = o(1), \qquad a \to 0. \tag{29}$$

Thus, one may neglect the second sum in (25), and write

$$E(x) = E_0(x) + \sum_{m=1}^{M} [\nabla_x g(x, x_m), Q_m] \tag{30}$$

with an error that tends to zero as $a \to 0$.

Let us estimate Q_m asymptotically, as $a \to 0$. Integrate equation (22) over S_m to get

$$Q_m = 2\int_{S_m} [f_e(s), N_s]ds - 2\int_{S_m} [N_s, B\sigma_m]ds. \tag{31}$$

We will show in the Appendix that the second term in the right-hand side of the above equation is equal to $-Q_m$ plus terms negligible compared with the first one as $a \to 0$. Thus,

$$Q_m = \int_{S_m} [f_e(s), N_s]ds.$$

Let us estimate the first term. It follows from equation (15) that

$$[N_s, f_e] = [N_s, E_e] - \frac{\zeta_m}{i\omega\mu_0}[N_s, [\nabla \times E_e, N_s]]. \tag{32}$$

If E_e tends to a finite limit as $a \to 0$, then formula (32) implies that

$$[N_s, f_e] = O(\zeta_m) = O\left(\frac{1}{a^\kappa}\right), \quad a \to 0. \tag{33}$$

By Lemma 2 the operator $(I - A_m)^{-1}$ is bounded, so $\sigma_m = O\left(\frac{1}{a^\kappa}\right)$, and

$$Q_m = O\left(a^{2-\kappa}\right), \quad a \to 0, \tag{34}$$

because integration over S_m adds factor $O(a^2)$. As $a \to 0$, the sum (30) converges to the integral

$$E = E_0 + \nabla \times \int_\Omega g(x,y)N(y)Q(y)dy, \tag{35}$$

where $Q(y)$ is the function such that

$$Q_m = Q(x_m)a^{2-\kappa}. \tag{36}$$

The function $Q(y)$ can be expressed in terms of E:

$$Q(y) = -\frac{8\pi}{3}h(y)i\omega\epsilon_0(\nabla \times E)(y), \tag{37}$$

see Appendix. Here the factor $\frac{8\pi}{3}$ appears if D_m are balls. Otherwise a tensorial factor c_m, depending on the shape of S_m, should be used in place of $\frac{8\pi}{3}$.

Thus, equation (35) takes the form

$$E(x) = E_0(x) - \frac{8\pi}{3}i\omega\epsilon_0\nabla \times \int_\Omega g(x,y)\nabla \times E(y)h(y)N(y)dy. \tag{38}$$

Let us derive physical conclusions from equation (38). Taking $\nabla \times \nabla \times$ of (38) yields

$$\begin{aligned}
\nabla \times \nabla \times E &= k^2E_0(x) \\
&\quad - \frac{8\pi}{3}i\omega\epsilon_0\nabla \times (\text{grad div} - \nabla^2)\int_\Omega g(x,y)\nabla \times E(y)h(y)N(y)dy \\
&= k^2E_0 - k^2\frac{8\pi}{3}i\omega\epsilon_0\nabla \times \int_\Omega g(x,y)\nabla \times E(y)h(y)N(y)dy \\
&\quad - \frac{8\pi}{3}i\omega\epsilon_0\nabla \times (\nabla \times E(x)h(x)N(x)) \\
&= k^2E(x) - \frac{8\pi}{3}i\omega\epsilon_0h(x)N(x)\nabla \times \nabla \times E \\
&\quad - \frac{8\pi}{3}i\omega\epsilon_0[\nabla(h(x)N(x)), \nabla \times E(x)].
\end{aligned} \tag{39}$$

Here we have used the known formula $\nabla \times \text{grad} = 0$, the known equation (10), and assumed for simplicity that $h(x)$ is a scalar function. It follows from (39) that

$$\nabla\times\nabla\times E = K^2(x)E - \frac{\frac{8\pi}{3}i\omega\epsilon_0}{1 + \frac{8\pi}{3}i\omega\epsilon_0h(x)N(x)}[\nabla(h(x)N(x)), \nabla\times E(x)], \tag{40}$$

where

$$K^2(x) = \frac{k^2}{1 + \frac{8\pi}{3} i\omega\epsilon_0 h(x) N(x)}, \quad k^2 = \omega^2 \epsilon_0 \mu_0. \tag{41}$$

If $\nabla \times E = i\omega\mu(x)H$ and $\nabla \times H = -i\omega\epsilon(x)E$, then

$$\nabla \times \nabla \times E = \omega^2 \epsilon(x)\mu(x)E + [\frac{\nabla \mu(x)}{\mu(x)}, \nabla \times E]. \tag{42}$$

Comparing this equation with (40), one can identify the last term in (40) as coming from a variable permeability $\mu(x)$. This $\mu(x)$ appears in the limiting medium due to the boundary currents on the surfaces S_m, $1 \leq m \leq M$. These currents appear because of the impedance boundary conditions (7). Let us identify the permeability $\mu(x)$. Denote $\Psi(x) := 1 + \frac{8\pi}{3} i\omega\epsilon_0 h(x) N(x)$. Let $\epsilon(x) = \epsilon_0$, $\epsilon_0 = const$, and define $\mu(x) := \frac{\mu_0}{\Psi(x)}$. Then $K^2 = \omega^2 \epsilon_0 \mu(x)$, and $\frac{\nabla \mu(x)}{\mu(x)} = -\frac{\nabla \Psi(x)}{\Psi(x)}$. Consequently, formula (40) has a clear physical meaning: the electromagnetic properties of the limiting medium are described by the variable permeability:

$$\mu(x) = \frac{\mu_0}{\Psi(x)} = \frac{\mu_0}{1 + \frac{8\pi}{3} i\omega\epsilon_0 h(x) N(x)}. \tag{43}$$

3 Conclusions

The limiting medium is described by the new refraction coefficient $K^2(x)$ *(see* (41)*) and the new term in the equation* (40)*. This term is due to the spatially inhomogeneous permeability* $\mu(x) = \frac{\mu_0}{\Psi(x)}$ *generated in the limiting medium by the boundary impedances. The field* $E(x)$ *in the limiting medium (and in equation* (40)*) solves equation* (38)*.*

Therefore, we predict theoretically the new physical phenomenon: by embedding many small particles with suitable boundary impedances into a given homogeneous medium, one can create a medium with a desired spatially inhomogeneous permeability (43)*.*

One can create material with a desired permeability $\mu(x)$ *by embedding small particles with suitably chosen boundary impedances. Indeed, by formula* (43) *one can choose a complex-valued, in general, function* $h(x)$*, and a non-negative function* $N(x) \geq 0$*, describing the density distribution of the small particles, so that the right-hand side of formula* (43) *will yield a desired function* $\mu(x)$*.*

4 Proofs of Lemmas

Proof of Lemma 1.
From equations (1) one derives (the bar stands for complex conjugate):

$$\int_{D_R} (\overline{H} \cdot \nabla \times E - E \cdot \nabla \times \overline{H}) dx = \int_{D_R} (i\omega\mu_0 |H|^2 - i\omega\epsilon_0 |E|^2) dx,$$

where $D_R := D \cap B_R$, and $R > 0$ is so large that $D_m \subset B_R := \{x \; : \; |x| \leq R\}$ for all m. Recall that $\nabla \cdot [E, \overline{H}] = \overline{H} \cdot \nabla \times E - E \cdot \nabla \times \overline{H}$. Applying the divergence theorem, using the radiation condition on the sphere $S_R = \partial B_R$, and taking real part, one gets

$$0 = \sum_{m=1}^{M} \mathrm{Re} \int_{S_m} [E, \overline{H}] \cdot N ds = \sum_{m=1}^{M} \mathrm{Re} \int_{S_m} \overline{\zeta_m}^{-1} \overline{E_t^-} \cdot E_t^- ds,$$

where E_t^- is the limiting value of E^t on S_m from D, $E^t = \zeta_m [H, N]$. This relation and assumption (4) imply $E_t^- = 0$ on S_m for all m. Thus, $E = H = 0$ in D.
Lemma 1 is proved. $\square$

Proof of Lemma 2.

If $\sigma_m = A_m \sigma_m$, then the functions $H = \frac{\nabla \times E}{i\omega\mu_0}$ and $E(x) = \nabla \times \int_{S_m} g(x,t)\sigma(t)dt$ solve equation (1) in D, E and H satisfy the radiation condition, and , condition (2). Thus, $E = H = 0$ in D. Consequently,

$$0 = \nabla \times \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt = (\mathrm{grad}\ \mathrm{div} - \nabla^2) \int_{S_m} g(x,t)\sigma_m(t)dt$$
$$= k^2 \int_{S_m} g(x,t)\sigma_m(t)dt, \quad x \in D.$$

This implies $\sigma_m(s) = 0$.
Lemma 2 is proved. $\square$

5 Appendix

In Section 5.1 equation (38) is derived. In Section 5.2 a linear algebraic system (LAS) is derived for finding vectors Q_m in equation (36).

5.1. Boundary condition (7) yields

$$0 = [N[E_e, N]] - \frac{\zeta_m}{i\omega\mu_0} [\nabla \times E_e, N] + [N, [\nabla \times \int_{S_m} g(s,t)\sigma_m(t)dt, N]]$$
$$- \frac{\zeta_m}{i\omega\mu_0} [\nabla \times \nabla \times \int_{S_m} g(x,s)\sigma_m(t)dt, N].$$

Let us denote

$$f_e := [N, [E_e, N]] - \frac{\zeta_m}{i\omega\mu_0}[\nabla \times E_e, N].$$

One has $\nabla \times \nabla \times = curlcurl = graddiv - \Delta$, and

$$\nabla_x \cdot \int_{S_m} g(x,t)\sigma_m(t)dt = -\int_{S_m} (\nabla_t g(x,t), \sigma_m(t))\, dt = \int_{S_m} g(x,t)\nabla_t \cdot \sigma_m(t)dt = 0,$$

and

$$-\nabla_x^2 \int_{S_m} g(x,t)\sigma_m(t)dt = k^2 \int_{S_m} g(x,t)\sigma_m(t)dt,$$

because $-\nabla_x^2 g(x,t) = k^2 g(x,t)$, $x \neq t$, see (10). Thus, using (13), one gets:

$$\begin{aligned} 0 = f_e + [\int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt, N_s] + \frac{1}{2}[\sigma_m(s), N_s] \\ + \frac{\zeta_m k^2}{i\omega\mu_0}[N_s, \int_{S_m} g(s,t)\sigma_m(t)dt]. \end{aligned}$$

Cross multiply this by N_s from the left and use the relation $N_s \cdot \sigma_m(s) = 0$, to obtain

$$\begin{aligned} 0 = [N_s, f_e] + [N_s, [\int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt, N_s]] + \frac{1}{2}\sigma_m(s) \\ - \zeta_m i\omega\epsilon_0 [N_s, [N_s, \int_{S_m} g(s,t)\sigma_m(t)dt]]. \end{aligned}$$

Note that

$$\begin{aligned} [N_s, [\int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt, N_s]] &= \int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt \\ &- N_s \int_{S_m} ([N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt, N_s), \\ &= \int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt. \end{aligned}$$

where the integral before the last one vanishes because its integrand vanishes as the dot product of two orthogonal vectors. Consequently,

$$\begin{aligned} \sigma_m(t) = 2[f_e(s), N_s] + 2\zeta_m i\omega\epsilon_0 [N_s, [N_s, \int_{S_m} g(s,t)\sigma_m(t)dt]] \\ - 2\int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt := A\sigma_m + f_m, \end{aligned}$$

which is equation (14), and

$$f_m := 2[f_e(s), N_s],$$

which is equation (15).

Denote

$$Q_m = \int_{S_m} \sigma_m(s) ds.$$

One has

$$\int_{S_m} [[N_s, [E_e(s), N_s]], N_s] ds = \int_{S_m} [E_e(s), N_s] ds = -\int_{D_m} \nabla_x \times E_e dx,$$

and

$$\int_{S_m} [[\nabla \times E_e, N_s], N_s] ds = -\left(\int_{S_m} \nabla \times E_e ds - \int_{S_m} N_s(\nabla \times E_e, N_s) ds \right)$$
$$= -\int_{S_m} \nabla \times E_e ds + \frac{4\pi a^2}{3} \nabla \times E_e(x_m)$$
$$= -\frac{8\pi a^2}{3} \nabla \times E_e(x_m), \qquad a \to 0.$$

Here we have used the formulas

$$\int_{S_m} \nabla \times E_e ds \sim 4\pi a^2 \nabla \times E_e(x_m), \quad a \to 0,$$

and

$$\int_S N_i N_j ds = \frac{4\pi a^2}{3} \delta_{ij},$$

where S is a sphere of radius a, $\{N_i\}_{i=1}^3$ are Cartesian components of the outer unit normal to the sphere S, and $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ii} = 1$.

Thus,

$$\int_{S_m} f_m(s) ds \sim -\frac{16\pi}{3} \zeta_m a^2 i\omega\epsilon_0 \nabla \times E_e(x_m) = O(a^{2-\kappa}),$$

as $a \to 0$, provided that

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \qquad 0 < \kappa < 1.$$

Let us now show that the term $\int_{S_m} A\sigma_m ds$ contributes $-Q$, so

$$Q_m \sim 0.5 \int_{S_m} f_m(s) ds, \qquad a \to 0. \tag{44}$$

One has

$$
\begin{aligned}
&-2\int_{S_m} ds\int_{S_m}[N_s,[\nabla_s g(s,t),\sigma_m(t)]]dt\\
&=-2\int_{S_m} ds\int_{S_m} dt\left(\nabla_s g(s,t)(N_s,\sigma_m(t))-\sigma_m(t)\frac{\partial g(s,t)}{\partial N_s}\right)dt\\
&=-2\int_{S_m} ds\int_{S_m} dt\nabla_s g(s,t)(N_s,\sigma_m(t))+\int_{S_m}\sigma_m(t)dt2\int_{S_m} ds\frac{\partial g(s,t)}{\partial N_s}.
\end{aligned}
$$

Since

$$2\int_{S_m} ds\frac{\partial g(s,t)}{\partial N_s}=-2\int_{D_m} dxk^2g(x,t)-1,$$

one gets

$$I:=\int_{S_m} dt\sigma_m(t)2\int_{S_m} ds\frac{\partial g(s,t)}{\partial N_s}=-\int_{S_m}\sigma_m(t)dt-2k^2\int_{S_m} dt\sigma_m(t)\int_{D_m} dxg(x,t).$$

Therefore

$$I:=-Q_m+I_1,$$

where the term I_1 is negligible compared with I.

If $\int_{S_m}|\sigma_m(t)|dt<\infty$ and $\int_{S_m}\sigma_m(t)dt\neq 0$, then

$$|\int_{S_m}\sigma_m(t)dt|\gg|\int_{S_m} dt\sigma_m(t)\int_{D_m} dxg(x,t)|,$$

because $|\int_{D_m} dxg(x,t)|=O(a^2)$ if $x\in D_m$.

One has

$$\big|-2\int_{S_m} ds\int_{S_m} dt\nabla_s g(s,t)(N_s,\sigma_m(t))\big|\ll\big|\int_{S_m}\sigma_m(t)dt\big|=|Q_m|,$$

because $|(N_s,\sigma_m(t))|=O(|s-t|)$ as $|s-t|\to 0$. Therefore,

$$Q_m=0.5\int_{S_m}\sigma_m(t)dt\sim-\frac{8\pi}{3}\zeta_m a^2 i\omega\epsilon_0\nabla\times E_e(x_m)ds,\quad a\to 0. \tag{45}$$

This yields the following formula (cf (30)):

$$E(x)=E_0(x)-\frac{8\pi}{3}i\omega\epsilon_0\sum_{m=1}^{M}\zeta_m a^2[\nabla g(x,x_m),\nabla\times E_e(x_m)],\qquad a\to 0. \tag{46}$$

or,

$$E(x) = E_0(x) - \frac{8\pi}{3} i\omega\epsilon_0 \sum_{m=1}^{M} h(x_m) a^{2-\kappa} \left[\nabla_x g(x, x_m), \nabla \times E_e(x_m)\right]. \quad (47)$$

Passing to the limit $a \to 0$, one obtains

$$E_e = E_0(x) - \frac{8\pi}{3} i\omega\epsilon_0 \int_\Omega [\nabla_x g(x, y), \nabla \times E_e(y)] h(y) N(y) dy, \quad (48)$$

where $h(x)$ is the function in the formula $\zeta_m = \frac{h(x_m)}{a^\kappa}$, and $N(x)$ is the function in the definition of $\mathcal{N}(\Delta)$. The above passage to the limit is done by Theorem 1 from [7], p. 206. It uses the convergence of the collocation method for solving equation (38), see [6]. Writing $E_e = E$ for the limiting field yields equation (38).

5.2. In this Section a numerical method is developed for solving many-body wave scattering problem when the scatterers are small in comparison with the wavelength. The method consists of a derivation of a linear algebraic system for finding vectors $\mathcal{P}_m := (\nabla \times E)(x_m)$, $1 \le m \le M$. If $\mathcal{P}_m$ are found, then by formulas (37) and (36) one finds

$$Q_m = -\frac{8\pi}{3} \pi i\omega\epsilon_0 h(x_m) a^{2-\kappa} \mathcal{P}_m, \quad (49)$$

and, by formula (30), the field $E(x)$.

Let us derive linear algebraic system for finding $\mathcal{P}_m$.

Apply $\nabla\times$ to equation (30), let $x = x_j$, $1 \le j \le M$, and replace $\sum_{m=1}^{M}$ by the sum $\sum_{m\neq j, m=1}^{M}$.

Then one obtains

$$\mathcal{P}_j = \mathcal{P}_{0j} - \frac{8\pi}{3} i\omega\epsilon_0 a^{2-\kappa} \sum_{m\neq j, m=1}^{M} (graddiv - \nabla^2) g(x, x_m)|_{x=x_j} h(x_m) \mathcal{P}_m, \quad (50)$$

where $1 \le j \le M$, and

$$\mathcal{P}_{0j} := (\nabla \times E_0)(x_j). \quad (51)$$

Equation (50) is a linear algebraic system for finding $\mathcal{P}_m$. In the above derivation we have used the formula

$$\nabla \times [A, B] = (B, \nabla)A - (A, \nabla)B + A(\nabla, B) - B(\nabla, A),$$

which yields

$$\nabla \times [\nabla g, Q_m] = (Q_m, \nabla)\nabla g - Q_m \nabla^2 g,$$

where

$$(Q_m, \nabla)\nabla g = \sum_{i=1}^{3} e_i \sum_{i'=1}^{3} Q_{mi'} \frac{\partial^2 g}{\partial x_i \partial x_{i'}}, \quad Q_{mi} = (Q_m, e_i),$$

and e_i, $i = 1, 2, 3$, is the orthonormal Cartesian basis of $\mathbb{R}^3$.

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